

# Fourth Quantization

Mir Faizal

Mathematical Institute, University of Oxford  
Oxford OX1 3LB, United Kingdom

## Abstract

In this paper we will analyse the creation of the multiverse. We will first calculate the wave function for the multiverse using third quantization. Then we will fourth quantize this theory. We will show that there is no single vacuum state for this theory. Thus, we can end up with a multiverse, even after starting from a vacuum state. This will be used as a possible explanation for the creation of the multiverse. We also analyse the effect of interactions in this fourth quantized theory.

## 1 Introduction

The existence of the multiverse first appeared in the many-worlds interpretation of quantum theory [1]. This idea has now resurfaced in the landscape of string theory [2, 3]. As this landscape is populated by  $10^{500}$  vacuum states [4], the possibility of all of them being real vacuum states for different universes remains an open one [5]. As a matter of fact, this model of the multiverse has also been used as an explanation for inflation in chaotic inflationary multiverse [6, 7, 8]. In fact, it is expected that the number of universes in the chaotic inflationary multiverse will be even more than the number of different string theory vacuum states. This is because for even for a single string theory vacuum state, the large-scale structure and the matter content in each of the locally Friedman parts will be different [9]. Thus, we should in principle be studding a collection of multiverse, each for a different string theory vacuum.

It may be noted that the big bang can be explained as the collision of two universes to form two new universes in the multiverse [10, 11]. Thus, to explain the cause of big bang we require a theory of many universes where these universes can interact with each other. Third quantization forms a natural formalism to study the such a model of the multiverse [12, 13]. This is because it is well know that we cannot explain the creation and annihilation of particles using a single particle quantum mechanics and we have to resort to a second quantized theory for that purpose. In doing so the Schrodinger equation can be treated as a classical field equation and then interaction terms added to it. Then when we quantize this theory, the creation and annihilation of particles can be explained. The Wheeler-DeWitt equation acts like the Schrodinger equation for quantum gravity. So, in analogy with the single particle quantum mechanics, we cannot explain the creation and annihilation of universes using a second quantized Wheeler-DeWitt equation, and we have to use a third quantized formalism for doing that purpose [15, 16]. In doing so the Wheeler-DeWitt equation is treated

as a classical field equation and interactions are added. These interaction terms cause the universes to get created and annihilated. The third quantization of the Brans-Dicke theories [17] and Kaluza-Klein theories [18] have been already thoroughly studied. Virtual black holes in third quantized  $f(R)$  gravity has also been studied [19]. In doing so a solution to the problem of time has also been proposed. The uncertainty principle for third quantization of  $f(R)$  gravity has also been analysed [20]. If we go to fourth quantization we will go to a theory of multiple multiverse. This seems to be a natural structure that will occur in chaotic inflationary multiverse because there will be a multiverse for each different string theory vacuum state.

In this paper will analyse a minisuperspace approximation to the Wheeler-DeWitt equation. We will first obtain an explicit expression for the wave function of the multiverse. Then we will analyse the creation of multiverse from a fourth vacuum state. Finally, we will analyse the effect of interaction of different multiverse. It may be noted just like in the third quantized formalism, the creation of universe can be explained, for the creation of the multiverse, we will have to quantize the third quantized theory one more time. Thus, in this paper fourth quantization is studied for the first time.

## 2 Third Quantization

For Einstein gravity, the Friedman-Robertson-Walker metric is given by

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2. \quad (1)$$

where  $d\Omega_3^2$  is the usual line element on the three sphere and  $N$  is the lapse function and here we have set  $k = 1$ . In this background, we have  $R_{ij} = 2\gamma_{ij}/a^2$  and  $R = 6/a^2$ . The Hamiltonian constraint in this minisuperspace approximation takes the following form,  $[\pi_a^2 - \omega^2(a)] = 0$ , where  $\omega^2(a) = -3\pi^2(3a^2 - \Lambda a^4)/4G$ . Now we promote the canonical momentum  $\pi_a$  to an operator, and so we have  $\pi_a = -i\partial_a$ . Thus, the Wheeler-DeWitt equation corresponding to this classical Hamiltonian constraint is given by

$$\left[ \frac{\partial^2}{\partial a^2} + \omega^2(a) \right] \phi[a] = 0. \quad (2)$$

We interpret Eq. (2), as a classical field equation of a classical field  $\phi(a)$ , whose classical action is given by

$$\mathcal{S}[\phi(a)] = \frac{1}{2} \int da \left( \left( \frac{\partial \phi}{\partial a} \right)^2 - \omega^2(a) \phi^2 \right). \quad (3)$$

Now obviously the variation of the third quantized action  $\mathcal{S}[\phi(a)]$  given by Eq. (3), will lead to the Wheeler-DeWitt equation (2). If we interpret the scaling factor  $a$  as the time variable, then the momentum conjugate to  $\phi(a)$  will be  $p = \dot{\phi}$ , where  $\dot{\phi}$  is the derivative of  $\phi$  with respect to  $a$ . Now the third quantized Hamiltonian obtained by the Legendre transformation, can be written as

$$\mathcal{H}(\phi(a), p(a)) = \frac{1}{2} p^2 + \frac{\omega^2(a)}{2} \phi^2. \quad (4)$$

This Hamiltonian given by Eq. (4) is the Hamiltonian for the harmonic oscillator with time-dependent frequency  $\omega(a)$  [21, 22]. Now we can write the third quantized Schrodinger for the multiverse as

$$\begin{aligned} O_0(a, \phi)\Phi(\phi, a) &= \left[ -i\frac{\partial}{\partial a} + \mathcal{H}(\phi(a), p(a)) \right] \Phi(\phi, a) \\ &= \left[ -i\frac{\partial}{\partial a} - \frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \frac{\omega^2(a)}{2}\phi^2 \right] \Phi(\phi, a) \\ &= 0. \end{aligned} \quad (5)$$

Here  $\Phi(\phi, a)$  is the wave function for the multiverse, and different modes correspond to different universes.

If we denote the two linearly independent solutions to the third quantized Schrodinger as  $u_0(a)$  and  $v_0(a)$ , and define  $\rho_0(a) = \sqrt{u_0^2 + v_0^2}$ , then we have

$$\frac{d^2}{da^2}\rho_0 + w_0^2(a)\rho_0 - \frac{\Omega_0^2}{\rho_0^3} = 0, \quad (6)$$

where  $\Omega_0$  is given by  $\Omega_0 = \dot{v}_0 u_0 - \dot{u}_0 v_0$ . Let the third quantized wave functions  $\Phi^s(\phi, a)$  be the solution to the following equation,

$$\begin{aligned} O_s(a, \phi)\Phi_0^s(\phi, a) &= \left[ -i\frac{\partial}{\partial a} + \mathcal{H}^s(\phi(a), p(a)) \right] \Phi^s(\phi, a) \\ &= \left[ -i\frac{\partial}{\partial a} - \frac{1}{2}\frac{\partial^2}{\partial \phi^2} + \frac{\phi^2(a)}{2} \right] \Phi^s(\phi, a) \\ &= 0. \end{aligned} \quad (7)$$

This is the equation of a simple harmonic oscillator system of unit mass and frequency. Here  $b$  denotes the time and  $\Phi^s(\phi, b)$  denote wave functions of this simple harmonic oscillator. This time  $b$  is related to  $a$  through the relation,  $d\rho_0^2 b = \Omega_0 da$ . Now we define  $U_{w0}(\rho_0, \Omega_0)$  as

$$U_{w0}(\rho_0, \Omega_0) = \exp\left(\frac{i\dot{\rho}_0}{2\rho_0}\phi^2\right) \exp\left[-\frac{i}{2}\left(\ln \frac{\rho_0}{\sqrt{\Omega_0}}\right)(\phi p + p\phi)\right], \quad (8)$$

where  $\Omega_0 U_{w0} O_s(b, \phi) U_{w0}^\dagger|_{b=b(a)} = \rho_0^2 O_0(a, \phi)$ . So, we can write  $\Phi(\phi, t) = U_{w0}\Phi^s|_{b=b(a)}$ . Furthermore, the wave function for the simple harmonic oscillator can be written as

$$\begin{aligned} \Phi^s(\phi, b)|_{b=b(a)} &= \frac{1}{\sqrt{2^n n! \sqrt{\dot{\Phi}}}} e^{-i(n+\frac{1}{2})b} \exp\left[-\frac{\phi^2}{2}\right] H_n(\phi)|_{b=b(a)} \\ &= \frac{1}{\sqrt{2^n n! \sqrt{\dot{\Phi}}}} \left(\frac{u_0(t) - iv_0(a)}{\rho_0(a)}\right)^{n+1/2} \\ &\quad \times \exp\left[-\frac{\phi^2}{2}\right] H_n(\phi), \end{aligned} \quad (9)$$

Now we define  $\delta_{u_1}(a)$  as  $2\dot{\delta}_{u_1} = w_0^2 u_1^2 - \dot{u}_1^2$  where  $u_1$  is a linear combination of  $u_0(a)$  and  $v_0(a)$ . Furthermore,  $U_f$  can be written as  $U_f = \exp[i(\dot{u}_1 \phi +$

$\delta_{u_1}(a)] \exp(-iu_1 p)$ , where  $U_f O_0 U_f^\dagger = O_0$ . Therefore, the wave functions  $\Phi(\phi, a)$  of the multiverse is given by

$$\begin{aligned} \Phi(\phi, a) &= U_f U_{w0} \Phi^s(\phi, b) |_{b=b(a)} \\ &= \frac{1}{\sqrt{2^n n! \rho_0(a)}} \left( \frac{\Omega_0}{\dot{\Phi}} \right)^{1/4} \left( \frac{u_0(t) - i v_0(t)}{\rho_0(a)} \right)^{n+1/2} \\ &\quad \times \exp[i(\dot{u}_1(a)\phi + \delta_{u_1}(a))] H_n \left( \sqrt{\Omega_0} \frac{\phi - u_1(a)}{\rho_0(a)} \right) \\ &\quad \times \exp \left[ \frac{(\phi - u_1(a))^2}{2} \left( -\frac{\Omega_0}{\rho_0^2(a)} + i \frac{\dot{\rho}_0}{\rho_0} \right) \right]. \end{aligned} \quad (10)$$

Thus, we have obtained the wave function for the full multiverse. The different modes here correspond to different universes.

### 3 Fourth Quantization

In order to analyse the creation of the multiverse from nothing, we need to construct a fourth quantized theory. Just as in second quantization, the first wave function was considered as a classical field, and in a third quantized formalism, the second quantized wave function was considered as a classical field, in a fourth quantized formalism the third quantized wave function will be considered as a classical field. So we start from defining the following symplectic product

$$(\Phi, \Phi') = -i \int d\phi [\Phi^*(\phi, a) \dot{\Phi}'^0(\phi, a) - \Phi'(\phi, a) \dot{\Phi}^{*0}(\phi, a)]. \quad (11)$$

Now we let  $\{\Phi_n\}$  and  $\{\Phi_n^*\}$  form a complete set of solutions to the third quantized Schrodinger equation for the multiverse, and suppose

$$(\Phi_n, \Phi_m) = M_{nm}, \quad (12)$$

$$(\Phi_n, \Phi_m^*) = 0, \quad (13)$$

$$(\Phi_n^*, \Phi_m^*) = -M_{nm}. \quad (14)$$

The condition given in Eq. (13) does not hold in general and is imposed here as a requirement on the complete set of solutions to the third quantized Schrodinger equation for the multiverse. We will also require  $M_{nm}$  to only have positive eigenvalues as this does not also hold in general. This matrix  $M_{nm}$  is Hermitian because

$$\begin{aligned} M_{nm} &= -i \int d\phi [\Phi_n^*(\phi, a) \dot{\Phi}_m^0(\phi, a) - \Phi_m(\phi, a) \dot{\Phi}_n^{*0}(\phi, a)] \\ &= \left[ -i \int d\phi [\Phi_m^*(\phi, a) \dot{\Phi}_n^0(\phi, a) - \Phi_n(\phi, a) \dot{\Phi}_m^{*0}(\phi, a)] \right]^* \\ &= M_{mn}^*. \end{aligned} \quad (15)$$

Now in analogy with second quantized quantum field theory, we promote  $\Phi$  and  $\dot{\Phi}^0$  to Hermitian operators, and impose the following,

$$\begin{aligned} [\Phi(\phi, a), \dot{\Phi}^0(\phi', a)] &= i\delta(\phi, \phi'), \\ [\Phi(\phi, a), \Phi(\phi', a)] &= 0, \\ [\dot{\Phi}^0(\phi, a), \dot{\Phi}^0(\phi', a)] &= 0, \end{aligned} \quad (16)$$

Now we can express  $\Phi(\phi, a)$  as,

$$\Phi(\phi, a) = \sum_n [a_n \Phi_n + a_n^\dagger \Phi_n^*]. \quad (17)$$

because  $\{\Phi_n\}$  and  $\{\Phi_n^*\}$  form a complete set of solutions to the third quantized Schrodinger equation for the multiverse. We define the fourth quantized vacuum state  $|0\rangle$ , as the state that is annihilated by  $a_n$ ,  $a_n|0\rangle = 0$ . This state is a purely mathematical object. It contains neither matter nor spacetime. However, we can create matter and spacetime from this vacuum state using  $a_n^\dagger$ . Thus, multiverse can be built from this vacuum state.

It may be noted that the division between  $\{\Phi_n\}$  and  $\{\Phi_n^*\}$  is not unique even after imposing conditions given by Eqs. (12)-(14). Due to this non-uniqueness in division between  $\{\Phi_n\}$  and  $\{\Phi_n^*\}$ , there is non-uniqueness in the definition of the fourth quantized vacuum state also. This can be seen by considering  $\{\Phi'_n\}$  and  $\{\Phi'^*_m\}$  as another complete set of solutions to the third quantized Schrodinger equation for the multiverse. Now we express  $\Phi(\phi, a)$  as

$$\Phi(\phi, a) = \sum_n [a'_n \Phi'_n + a'^{\dagger}_n \Phi'^*_n]. \quad (18)$$

Here  $\{\Phi'_n\}$  and  $\{\Phi'^*_m\}$  also satisfying conditions similar to the conditions given in Eqs. (12)-(14). We define a corresponding fourth quantized vacuum state  $|0'\rangle$  as the state annihilated by  $a'_n$ ,  $a'_n|0'\rangle = 0$ . The spacetime and matter can again be built by repeated action of  $a'^{\dagger}_n$  on  $|0'\rangle$ . As  $\Phi_n$  and  $\Phi_n^*$  form a complete set of solutions to the third quantized Schrodinger equation for the multiverse, we can express  $\Phi'_n$  as a linear combination of  $\Phi_n$  and  $\Phi_n^*$ ,

$$\Phi'_n = \sum_m [\alpha_{nm} \Phi_m + \beta_{nm} \Phi_m^*]. \quad (19)$$

Thus, we have

$$a_n = \sum_m [\alpha_{nm} a'_m + \beta_{nm}^* a'^{\dagger}_m], \quad (20)$$

$$a_n^\dagger = \sum_m [\alpha_{nm}^* a'^{\dagger}_m + \beta_{nm} a'_m]. \quad (21)$$

As long as  $\beta_{nm} \neq 0$ , these two Fock spaces based on different complete set of solutions to third quantized Schrodinger equation for the multiverse are different. So,  $a_n|0\rangle = 0$ , however

$$\begin{aligned} a_n|0'\rangle &= \sum_m [\alpha_{nm} a'_m + \beta_{nm}^* a'^{\dagger}_m]|0'\rangle \\ &= \sum_m \beta_{nm}^* a'^{\dagger}_m|0'\rangle \neq 0. \end{aligned} \quad (22)$$

So,  $a_n|0'\rangle$  is a multiverse state given by

$$\langle 0'|a_n^\dagger a_n|0'\rangle = \sum_m \sum_k \beta_{nm} \beta_{nk}^* M_{mk}. \quad (23)$$

Thus, even though we started from a vacuum state, we ended up with the multiverse.

## 4 Interactions

In the previous section we analysed how the multiverse. In this section we will analyse the effect of interaction in the multiverse. To analyse the interaction of these universes we first define a fourth quantized Lagrangian as

$$S = \int d\phi da \Phi(\phi, a) O_0(a, \phi) \Phi(\phi, a). \quad (24)$$

Now we can add interaction terms  $S_I[\Phi(\phi, a)]$  to it,

$$S_T[\Phi(\phi, a)] = S[\Phi(\phi, a)] + S_I[\Phi(\phi, a)]. \quad (25)$$

In order to analyse the perturbation theory of these universes, we first define

$$J\Phi = \int d\phi da J(\phi, a) \Phi(\phi, a) \quad (26)$$

and then let  $a \rightarrow ia$ . Now we can the Euclidean partition function as

$$Z[J] = \int D\Phi \exp -\mathcal{S}[\Phi]_{ET} + J\Phi, \quad (27)$$

where  $\mathcal{S}[\Phi]_{ET}$  is the Euclidean version of  $\mathcal{S}[\Phi]_T$  obtained by letting  $a \rightarrow ia$ . Now we can define

$$W[J] = -i \ln Z[J]. \quad (28)$$

We can also define the effective action as

$$\Gamma[\Phi_b] = W[J] - \Phi_b J, \quad (29)$$

where

$$\Phi_b = \frac{\delta W[J]}{\delta J}. \quad (30)$$

The full quantum equation of motion will be given by

$$\frac{\delta \Gamma}{\delta \Phi_b} = 0. \quad (31)$$

We will analyse a simple interaction term given by

$$\mathcal{S}_{EI} = \frac{\lambda}{4!} \int d\phi da \Phi^4(\phi, a). \quad (32)$$

If we represent matter and gauge fields collectively by  $\chi$  and include the contribution from  $\chi$  in our formalism, then  $\phi$  would also depend on  $\chi$ , and so,  $\Phi$  would also depend on  $\chi$ . Thus, the collision of multiverse  $M_1$  and  $M_2$  into two universes  $M_3$  and  $M_4$ . Let  $B$  be a third quantized charge that remains conserved when two universes collide [23]. Also let the total number of universes (with a positive value of  $B$ ) and anti-universes (with a negative value of  $B$ ) in multiverse  $M_k$  (with  $k = 1, 2, 3, 4$ ) be  $n_k$  and  $m_k$ , respectively. Now if the multiverse  $M_1$  and  $M_2$  have formed from nothing without violating the conservation of  $B$ , then we have,  $Bn_1 - Bm_1 + Bn_2 - Bm_2 = 0$ . Here  $Bn_1 + Bn_2$  represent the total  $B$  number of the universes and  $-Bm_1 - Bm_2$  represent the total  $B$  number of the anti-universes. Furthermore, after the collision, if the  $B$

number is conserved, we will have,  $Bn_3 - Bm_3 + Bn_4 - Bm_4 = 0$ . However the  $B$  number in the multiverse  $M_3$  or  $M_4$  need not be separately conserved, so we have,  $Bn_3 - Bm_3 \neq 0$ , and  $Bn_4 - Bm_4 \neq 0$ . Thus, after the collision some multiverse can have more positive  $B$  than if the other multiverse has more negative  $B$ . Previously, it was proposed that an initial universe broke into two universes and this was proposed as a explanation for the dominance of matter over anti-matter in our universe [11]. In this paper this model is generalized to include the full multiverse.

## 5 Conclusion

In this paper we analysed third quantization of Wheeler-deWitt equation for a Friedman-Robertson-Walker universe with a cosmological constant. We analyse the multiverse in this formalism and calculated the wave function of the multiverse. We also constructed the Fock space for the multiverse. As there was no single vacuum state, we can ended up with a multiverse state, even after starting from a vacuum state. This was used as an possible explanation of the creation of multiverse. We also see the effect of interactions of multiverse.

It is know that in second quantized quantum field theory, we can construct conserved charges which remain conserved, even when the particle number is not conserved. These conserved charges are generated by the invariance of the second quantized Lagrangian under some symmetry. In analogy with second quantized quantum field theory, we can also construct a third quantized Noether's theorem. Thus, if there is a symmetry which will leave a third quantized Lagrangian invariant, then the charge corresponding to it will be conserved even if the number of universes is not conserved. Here we argued that a third quantized Noether's charge will not remain conserved for individual multiverse, in case of a fourth quantized interacting theory. It will be interesting to construct third quantized Noether's charges for different quantum cosmological models and analyse the conservation of these Noether's charges in a fourth quantized theory.

It may be noted that canonical quantum gravity has evolved into loop quantum gravity [24, 25, 26, 27]. Third quantization of loop quantum gravity has led to the development of group field theory [28, 29, 30, 31]. Recently, group field cosmology has also been developed [32, 33]. It will be interesting to analyse the creation of universe from nothing using group field cosmology. Furthermore, the group field cosmology has also been supersymmetrized [34]. In this supergroup field cosmology, there are universes with both bosonic and fermionic distributions in the multiverse. It is hoped that by considering a supersymmetric distribution of universes, we might get better understanding of the cosmological constant. Thus, it will be interesting to supersymmetric this present work and derive an explicit expression for the multiverse containing both bosonic and fermionic universes.

## References

- [1] is the H. Everett, Rev. Mod. Phys. 29, 454 (1957)
- [2] S. K. Ashok and M. R. Douglas, JHEP. 01, 060 (2004)

- [3] P. C. W. Davies, *Mod. Phys. Lett. A*19, 727 (2004)
- [4] F. Denef and M. R. Douglas, *JHEP.* 0405, 072 (2004)
- [5] L. M. Houghton and R. Holman, *JCAP.* 0902, 006 (2009)
- [6] A. Linde, *Mod. Phys. Lett. A*1, 81 (1986)
- [7] A. Linde, *JCAP.* 0701, 022 (2007)
- [8] M. M. Cirkovic and N. Bostrom, *Astrophys. Space Sci.* 274, 675 (2000)
- [9] A. Linde and V. Vanchurin, *Phys. Rev. D*81, 083525 (2010)
- [10] P. J. Steinhardt and N. Turok, *New Astron. Rev.* 49, 43 (2005)
- [11] M. Faizal, *Mod. Phys. Lett. A*27, 1250007 (2012)
- [12] S. R. Perez and P. F. G. Diaz, *Phys. Rev. D*81, 083529 (2010)
- [13] S. R. Perez, Y. Hassouni and P. F. G. Diaz, *Phys. Lett. B* 683, 1 (2010)
- [14] S. W. Hawking, *Phys. Rev. Lett.* B195, 337 (1987)
- [15] C. Teitelboim, *Phys. Rev. D*25, 3159 (1982)
- [16] K. Kueha, *J. Math. Phys.* 22, 2640 (1981)
- [17] L. O. Pimentel and C. Mora, *Phys. Lett. A*280, 191 (2001)
- [18] Y. Ohkuwa, *Int. J. Mod. Phys. A*13, 4091 (1998)
- [19] M. Faizal, *JETP.* 114, 400 (2012)
- [20] Y. Ohkuwa and Y. Ezawa, *Class. Quantum Grav.* 29, 215004 (2012)
- [21] L. S. Brown, *Phys. Rev. Lett.* 18, 510 (1967)
- [22] D. Y. Song, *Phys. Rev. A* 62, 014103 (2000)
- [23] M. Faizal, *arXiv:1303.5478*
- [24] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge, UK 2007
- [25] G. Date and G. M. Hossain, *SIGMA* 8, 010 (2012)
- [26] A. Ashtekar and A. Corichi, *Class. Quant. Grav.* 20, 4473 (2003)
- [27] A. Ashtekar, *Nature Physics* 2, 726 (2006)
- [28] A. Baratin and D. Oriti, *Phys. Rev. D* 85, 044003 (2012)
- [29] M. Smerlak, *Class. Quant. Grav.* 28, 178001 (2011)
- [30] A. Baratin, F. Girelli and D. Oriti, *Phys. Rev. D*83, 104051 (2011)
- [31] A. Tanasa, *J. Phys. A*45, 165401 (2012)



- [32] G. Calcagni, S. Gielen and D. Oriti, *Class. Quantum Grav.* 29, 105005 (2012)
- [33] M. Faizal, [arXiv:1301.0224](#)
- [34] M. Faizal, *Class. Quant. Grav.* 29, 215009 (2012)